DIRECT METHODS OF SOLVING LINEAR REVERSE HEAT-CONDUCTION PROBLEMS

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Approximate analytical methods are proposed for solving reverse heat-conduction problems for the case of a semiinfinitely large body and a plane layer with movable or stationary boundaries. The applicability limits of the results are evaluated.

Reverse heat-conduction problems of the first kind are those where the thermal flux or the surface temperature are to be determined from the known temperature inside the body. Despite the inherent instability of such problems, in certain cases they can be solved by direct methods. We propose here to determine the transient thermal fluxes by semianalytical methods based on the solution of integral equations.

We consider the second boundary-value problem of heat conduction for the case of a semiinfinitely large body with a movable boundary:

$$C\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right),\ x>X(\tau),\ \tau>0, \tag{1}$$

$$\left.\begin{aligned} &T(x,\ 0)=T_0,\\ &\lambda\frac{\partial T(X(\tau),\ \tau)}{\partial x}+q(\tau)=0,\\ &\frac{\partial T(\infty,\ \tau)}{\partial x}=T(\infty,\ \tau)=0. \end{aligned}\right\} \tag{2}$$

With the aid of the Kirchhoff transformation $\theta=(1/\lambda_0)\int\limits_0^T\lambda(T)dT$, where λ_0 denotes some constant, problem (1)-(2) can be reduced to the form

$$\frac{\partial\theta}{\partial\tau}=a\frac{\partial^2\theta}{\partial x^2},\ x>X(\tau),\ \tau>0, \tag{3}$$

$$\left.\begin{aligned} &\theta(x,\ 0)=\theta_0,\\ &\lambda_0\frac{\partial\theta(X(\tau),\ \tau)}{\partial x}+q(\tau)=0,\\ &\frac{\partial\theta(\infty,\ \tau)}{\partial x}=\theta(\infty,\ \tau)=0. \end{aligned}\right\} \tag{4}$$

We will now assume that a = const. This condition is either approximately or accurately enough satisfied for certain metals and nonmetallic materials.

The solution to (3)-(4) will be expressed in terms of the thermal potential in a simple layer:

$$\vartheta(x,\ \tau)\equiv\theta(x,\ \tau)-\theta_0=\frac{a}{2\sqrt{\pi}}\int\limits_0^\tau\nu(\xi)\frac{\exp\left[-\frac{(x-X(\xi))^2}{4a(\tau-\xi)}\right]}{\sqrt{a(\tau-\xi)}}d\xi. \tag{5}$$

The limit value obtained for the derivative of the thermal potential, as the boundary of the region is approached from inside, will be determined according to the formula for a temperature jump [1], which

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in this case reduces to

$$\frac{\partial\theta(X+0,\,\tau)}{\partial x}=\frac{\partial\theta(X,\,\tau)}{\partial x}-\frac{\nu(\tau)}{2},$$

where

$$\frac{\partial\theta(X,\,\tau)}{\partial x}=-\frac{1}{4\sqrt{a\pi}}\int\limits_0^\tau \nu(\xi)\frac{X(\tau)-X(\xi)}{(\tau-\xi)^{3/2}}\exp\left[-\frac{(X(\tau)-X(\xi))^2}{4a(\tau-\xi)}\right]d\xi.$$

Furthermore, noting that $-\lambda_0 = \partial\theta(X + 0,\ \tau)/\partial x = q(\tau)$, we have

$$q(\tau)=\lambda_0\frac{\nu(\tau)}{2}+\frac{\lambda_0}{2}\int\limits_0^\tau \nu(\xi)\frac{X(\tau)-X(\xi)}{2\sqrt{a\pi(\tau-\xi)^3}}\exp\left[-\frac{(X(\tau)-X(\xi))^2}{4a(\tau-\xi)}\right]d\xi. \tag{6}$$

Let us approximate this expression. The heating period $[0,\ \tau]$ will be divided into n generally unequal intervals ($n = 1, 2, \ldots, m$) and curves $\nu(\tau)$ and $X(\tau)$ will be replaced by a staircase so as to satisfy the conditions $\bar\nu_i = (\nu_i + \nu_{i-1})/2$, $\bar X_i = (X_i + X_{i-1})/2$ on each segment. We then obtain the following expression for $q(\tau)$:

$$q(\tau_n)\simeq\frac{\lambda_0}{2}\left\{\bar\nu_n-\sum_{i=1}^{n}\bar\nu_i\int\limits_{\tau_{i-1}}^{\tau_i}\frac{\partial}{\partial\tau}\Phi\left[\frac{\bar X_n-\bar X_i}{2\sqrt{a(\tau_n-\xi)}}\right]d\xi\right\}$$

$$=\frac{\lambda_0}{2}\left\{\bar\nu_n+\sum_{i=1}^{n}\bar\nu_i\Phi\left[\frac{\bar X_n-\bar X_i}{2\sqrt{a(\tau_n-\xi)}}\right]_{\xi=\tau_{i-1}}^{\xi=\tau_i}\right\},$$

where $\Phi[u] = (2/\sqrt{\pi})\int\limits_0^u e^{-\eta^2}d\eta$.

Noting that the limit

$$\lim_{i\to n}\Phi\left[\frac{\bar X_n-\bar X_i}{2\sqrt{a(\tau_n-\tau_i)}}\right]=0,$$

we write the relation for the thermal flux as

$$q_n=\frac{\lambda_0}{2}\left\{\bar\nu_n+\sum_{i=1}^{n-1}\bar\nu_i\left(\Phi\left[\frac{\bar X_n-\bar X_i}{2\sqrt{a(\tau_n-\tau_i)}}\right]-\Phi\left[\frac{\bar X_n-\bar X_i}{2\sqrt{a(\tau_n-\tau_{i-1})}}\right]\right)\right\}. \tag{7}$$

The values of the thermal potential density ν on the given time grid must be determined according to (5) from the known temperature $\theta(x_1,\ \tau)$ at point $x = x_1$. An analogous approximation of (5) yields

$$\vartheta_n(x_1)=\sqrt{a}\sum_{i=1}^{n}\bar\nu_i\varphi_i^n, \tag{8}$$

where

$$\varphi_i^n=\sqrt{\tau_n-\tau_{i-1}}\,i\Phi^*\left[\frac{x-\bar X_i}{2\sqrt{a(\tau_n-\tau_{i-1})}}\right]-\sqrt{\tau_n-\tau_i}\,i\Phi^*\left[\frac{x-\bar X_i}{2\sqrt{a(\tau_n-\tau_i)}}\right],$$

$$i\Phi^*[u]=\frac{1}{\sqrt{\pi}}\exp[-u^2]-u[1-\Phi(u)].$$

From (8) follows the recurrence relation between $\bar\nu_n$ and $\bar\nu_i$:

$$\bar\nu_n=\frac{1}{\varphi_n^n}\left[\frac{1}{\sqrt{a}}\vartheta_n(x_1)-\sum_{i=1}^{n-1}\bar\nu_i\varphi_i^n\right]. \tag{9}$$

In the case of a stationary boundary, the determination of thermal fluxes becomes simpler. Omitting here all intermediate transformations, we show the final result for a semiinfinitely large body with $X(\tau) = 0$ [2]:

1) with a staircase approximation of $q(\tau)$†

$$\bar q_n=\frac{1}{f_n^n}\left[\frac{\lambda_0}{2\sqrt{a}}\vartheta_n(x_1)-\sum_{i=1}^{n-1}\bar q_i f_i^n\right],$$

†Analogous expressions have been obtained in [3] and [4].

2) with a piecewise-linear approximation of $q(\tau)$

$$q_n = \frac{\tau_n - \tau_{n-1}}{F_n^n}\left[\frac{\lambda_0}{2\sqrt{a}}\vartheta_n(x_1) + q_{n-1}\left(f_n^n + \frac{F_n^n}{\tau_n - \tau_{n-1}}\right) - \sum_{i=1}^{n-1}\left(F_i^n k_i - f_i^n q_{i-1}\right)\right], \tag{10}$$

where

$$f_i^n = \sqrt{\tau_n - \xi}\, i\Phi^*\left[\frac{x_1}{2\sqrt{a(\tau_n - \xi)}}\right]_{\xi=\tau_{i-1}}^{\xi=\tau_i},$$

$$F_i^n = \left(\tau_{i-1} - \tau_n - \frac{x_1^2}{6a}\right) f_i^n + \frac{1}{3\sqrt{\pi}}(\tau_n - \xi)^{\frac{3}{2}} \exp\left[\frac{-x_1^2}{4a(\tau_n - \xi)}\right]_{\xi=\tau_{i-1}}^{\xi=\tau_i}, \tag{11}$$

$$k_i = \frac{q_i - q_{i-1}}{\tau_i - \tau_{i-1}}.$$

We next consider a plate which is heated (or cooled) on two sides with thermal fluxes $q(\tau)$ and $q_{in}(\tau)$, respectively, both varying with time. One boundary of this plate is movable and its displacement mode $X(\tau)$ is assumed known. The initial temperature distribution is constant and equal to T_0. The temperature distribution in the model is in this case described by the following system of equations:

$$\left.\begin{aligned}
&\frac{\partial\theta}{\partial\tau} = a\frac{\partial^2\theta}{\partial x^2},\quad X(\tau) < x < b_0,\ \tau > 0,\\
&\theta(x, 0) = \theta_0,\\
&\lambda_0 \frac{\partial\theta(X(\tau), \tau)}{\partial x} + q(\tau) = 0,\\
&\lambda_0 \frac{\partial\theta(b_0, \tau)}{\partial x} + q_{in}(\tau) = 0.
\end{aligned}\right| \tag{12}$$

The solution to the forward heat-conduction problem is sought as the sum of thermal potentials in a simple layer:

$$\vartheta(x, \tau) \equiv \theta(x, \tau) - \theta_0 = \frac{a}{2\sqrt{\pi}}\left\{\int_0^\tau \nu_1(\xi)\frac{\exp\left[-\frac{(x - X(\xi))^2}{4a(\tau-\xi)}\right]}{\sqrt{a(\tau-\xi)}}d\xi + \int_0^\tau \nu_2(\xi)\frac{\exp\left[-\frac{(b_0 - x)^2}{4a(\tau-\xi)}\right]}{\sqrt{a(\tau-\xi)}}d\xi\right\}.$$

For the thermal potential densities ν_1 and ν_2 corresponding to the two respective plate boundaries we write the following system of integral equations

$$q(\tau) = \frac{\lambda_0}{2}\left\{\frac{\nu_1(\tau)}{2} + \int_0^\tau \nu_1(\xi)\frac{X(\tau) - X(\xi)}{2\sqrt{a\pi(\tau-\xi)^3}}\exp\left[-\frac{(X(\tau) - X(\xi))^2}{4a(\tau-\xi)}\right]d\xi\right. \cdot$$

$$\left. - \int_0^\tau \nu_2(\xi)\frac{b_0 - X(\tau)}{2\sqrt{a\pi(\tau-\xi)^3}}\exp\left[-\frac{(b_0 - X(\tau))^2}{4a(\tau-\xi)}\right]d\xi\right\},$$

$$q_{in}(\tau) = \frac{\lambda_0}{2}\left\{-\frac{\nu_2(\tau)}{2} + \int_0^\tau \nu_1(\xi)\frac{b_0 - X(\xi)}{2\sqrt{a\pi(\tau-\xi)^3}}\exp\left[-\frac{(b_0 - X(\xi))^2}{4a(\tau-\xi)}\right]d\xi\right\}.$$

We now approximate $q(\tau)$ and $q_{in}(\tau)$ in the same manner as before. As a result, we obtain

$$q_n = \frac{\lambda_0}{2}\left\{\bar{\nu}_{1n} + \sum_{i=1}^{n-1}\bar{\nu}_{1i}\Phi\left[\frac{\bar{X}_n - \bar{X}_i}{2\sqrt{a(\tau_n-\xi)}}\right] - \sum_{i=1}^{n}\bar{\nu}_{2i}\Phi\left[\frac{b_0 - \bar{X}_n}{2\sqrt{a(\tau_n-\xi)}}\right]\right\}_{\xi=\tau_{i-1}}^{\xi=\tau_i}, \tag{13}$$

$$q_{in\,n} = \frac{\lambda_0}{2}\left\{-\bar{\nu}_{2n} + \sum_{i=1}^{n}\bar{\nu}_{1i}\Phi\left[\frac{b_0 - \bar{X}_i}{2\sqrt{a(\tau_n-\xi)}}\right]_{\xi=\tau_{i-1}}^{\xi=\tau_i}\right\}, \tag{14}$$

where

$$\bar{\nu}_i = \frac{\nu_i + \nu_{i-1}}{2};\quad \bar{X}_i = \frac{X_i + X_{i-1}}{2}.$$

The corresponding expression for the model temperature at the time τ_n is

$$\vartheta_n(x) = -\sqrt{a}\left\{\sum_{i=1}^{n} \bar{\nu}_{1i}\varphi_{1i}^n + \sum_{i=1}^{n} \bar{\nu}_{2i}\varphi_{2i}^n\right\}, \tag{15}$$

where

$$\varphi_{1i}^n = \sqrt{\tau_n - \xi}\, i\Phi^*\left[\frac{x - X_i}{2\sqrt{a(\tau_n - \xi)}}\right]_{\xi=\tau_{i-1}}^{\xi=\tau_i},$$

$$\varphi_{2i}^n = \sqrt{\tau_n - \xi}\, i\Phi^*\left[\frac{b_0 - x}{2\sqrt{a(\tau_n - \xi)}}\right]_{\xi=\tau_{i-1}}^{\xi=\tau_i}.$$

We now assume that the relations $T(x_1, \tau)$ and $T(x_2, \tau)$ are known at two points of the plate. With the aid of expression (15), it is then possible to write an algebraic system of equations for the thermal potential densities ν_1 and ν_2:

$$\begin{aligned} \bar{\nu}_{1n}\varphi_{1n}^n\big|_{x_1} + \bar{\nu}_{2n}\varphi_{2n}^n\big|_{x_1} &= \frac{\vartheta_n(x_1)}{\sqrt{a}} - \sum_{i=1}^{n-1}\left(\bar{\nu}_{1i}\varphi_{1i}^n\big|_{x_1} - \bar{\nu}_{2i}\varphi_{2i}^n\big|_{x_1}\right), \\ \bar{\nu}_{1n}\varphi_{1n}^n\big|_{x_2} + \bar{\nu}_{2n}\varphi_{2n}^n\big|_{x_2} &= \frac{\vartheta_n(x_2)}{\sqrt{a}} - \sum_{i=1}^{n-1}\left(\bar{\nu}_{1i}\varphi_{1i}^n\big|_{x_2} - \bar{\nu}_{2i}\varphi_{2i}^n\big|_{x_2}\right). \end{aligned} \tag{16}$$

We change from variables $\bar{\nu}_1$ and $\bar{\nu}_2$ to thermal fluxes q and q_{in} according to formulas (13) and (14).

If both plate boundaries are stationary, then the unknown thermal fluxes q and q_{in} are found (as can be easily proved) as the solution to the following system of equations:

$$\begin{aligned} &\bar{q}_n \tilde{\vartheta}(x_1, \tau_n - \tau_{n-1}) + \bar{q}_{\mathrm{in}_n} \tilde{\vartheta}(b - x_1, \tau_n - \tau_{n-1}) \\ &= \sum_{i=1}^{n-1}\{\bar{q}_i \tilde{\vartheta}(x_1, \tau_n - \xi) + \bar{q}_{\mathrm{in}_i} \tilde{\vartheta}(b - x_1, \tau_n - \xi)\}_{\xi=\tau_{i-1}}^{\xi=\tau_i} - \vartheta_n(x_1), \\ &\bar{q}_n \tilde{\vartheta}(x_2, \tau_n - \tau_{n-1}) + \bar{q}_{\mathrm{in}_n} \tilde{\vartheta}(b - x_2, \tau_n - \tau_{n-1}) \\ &= \sum_{i=1}^{n-1}\{\bar{q}_i \tilde{\vartheta}(x_2, \tau_n - \xi) + \bar{q}_{\mathrm{in}_i} \tilde{\vartheta}(b - x_2, \tau_n - \xi)\}_{\xi=\tau_{i-1}}^{\xi=\tau_i} - \vartheta_n(x_2), \end{aligned} \tag{17}$$

where (cf. [5])

$$\tilde{\vartheta}(x, \tau) = \frac{2\sqrt{a\tau}}{\lambda_0}\sum_{j=0}^{\infty}\left\{i\Phi^*\left[\frac{2jb + x}{2\sqrt{a\tau}}\right] + i\Phi^*\left[\frac{2b(j+1) - x}{2\sqrt{a\tau}}\right]\right\}.$$

When the back surface is thermally insulated, then the sought solution becomes simpler:

$$\bar{q}_n = \frac{1}{\tilde{\vartheta}(x_1, \tau_n - \tau_{n-1})}\left\{\sum_{i=1}^{n-1} \bar{q}_i[\tilde{\vartheta}(x_1, \tau_n - \tau_i) - \tilde{\vartheta}(x_1, \tau_n - \tau_{i-1})] - \vartheta_n(x_1)\right\} \tag{18}$$

and the temperature at one point $x = x_1$ needs to be known.

In practice it is often easier to measure the temperature of a thermally insulated surface. In this case it is possible to obtain formulas not containing infinite series. Of basic importance in constructing the algorithm here is the introduction of an auxiliary function $g(\tau)$ which would be related to $q(\tau)$ in a definite way through a continuous operator and which would be defined by a simple integral equation [6].

We select function $g(\tau)$ defined by an integral equation analogous in form to Eq. (5). In the case of an infinitely large plate it will then be possible to calculate the thermal flux by the main part of the algorithm which had been designed for a semiinfinitely large body. We consider the following problem:

$$\frac{\partial\theta}{\partial\tau} = a\frac{\partial^2\theta}{\partial x^2}, \quad \theta(x, 0) = 0,$$

$$\theta(b, \tau) = \theta_w(\tau), \quad \frac{\partial\theta(0, \tau)}{\partial x} = 0.$$

We then perform the Laplace transformation:

$$s\bar{\theta}(x,\ s) = a\ \frac{d^2\bar{\theta}(x,\ s)}{dx^2}, \tag{19}$$

$$\bar{\theta}(b,\ s) = \bar{\theta}_w,\quad \frac{d\bar{\theta}(0,\ s)}{dx} = 0,$$

where

$$L[\theta(x,\ \tau)] = \int_0^\infty \theta(x,\ \tau)\exp(-s\tau)\,d\tau = \bar{\theta}(x,\ s).$$

The solution to this problem is

$$\bar{\theta}(x,\ s) = \bar{\theta}_w\ \frac{\operatorname{ch}\left(\sqrt{\frac{s}{a}}\,x\right)}{\operatorname{ch}\left(\sqrt{\frac{s}{a}}\,b\right)}. \tag{20}$$

The transform expression for the thermal flux is

$$\bar{q}(s) = \lambda_0 \frac{d\bar{\theta}(x,\ s)}{dx}\bigg|_{x=b} = \lambda_0\sqrt{\frac{s}{a}}\,\bar{\theta}_w\ \frac{\operatorname{sh}\left(b\sqrt{\frac{s}{a}}\right)}{\operatorname{ch}\left(b\sqrt{\frac{s}{a}}\right)}. \tag{21}$$

From (20) and (21) follows

$$\bar{q}(s) = \lambda_0\sqrt{\frac{s}{a}}\,\bar{\theta}(0,\ s)\operatorname{sh}\left(b\sqrt{\frac{s}{a}}\right).$$

We define the transformation of function $g(\tau)$ as follows

$$\bar{g}(s) = \lambda_0\bar{\theta}(0,\ s)\, s\sqrt{\frac{s}{a}}\exp\left(b\sqrt{\frac{s}{a}}\right). \tag{22}$$

Then

$$\bar{q}(s) = \frac{\bar{g}(s)}{s}\operatorname{sh}\left(b\sqrt{\frac{s}{a}}\right)\exp\left(-b\sqrt{\frac{s}{a}}\right) = \frac{\bar{g}(s)}{2s}\left[1-\exp\left(-2b\sqrt{\frac{s}{a}}\right)\right]. \tag{23}$$

Reverting to the originals (22) and (23), we obtain an integral equation for $g(\tau)$ as well as an equation which relates functions $q(\tau)$ and $g(\tau)$:

$$q(\tau) = \frac{1}{2}\int_0^\tau g(\xi)\left[1 - i\Phi^*\left(\frac{b}{\sqrt{a(\tau-\xi)}}\right)\right]d\xi, \tag{24}$$

$$\frac{d\theta(0,\ \tau)}{d\tau} = \frac{1}{\lambda_0}\sqrt{\frac{a}{\pi}}\int_0^\tau g(\xi)\ \frac{\exp\left[-\frac{b^2}{4a(\tau-\xi)}\right]}{\sqrt{(\tau-\xi)}}\,d\xi. \tag{25}$$

Equation (25) is identical in form to the integral equation for the thermal flux transmitted to a semi-infinitely large body with a movable boundary. If the derivative $d\theta(0,\ \tau)/d\tau$ is known, then determining the function $g(\tau)$ is completely analogous here to determining the function $q(\tau)$ from the known temperature $\theta(b,\ \tau)$. For instance, with the staircase approximation of $g(\tau)$ we obtain

$$\bar{g}(\tau_n) = \frac{1}{f_n^n}\left[\frac{\lambda_0}{2\sqrt{a}}\ \frac{d\theta(0,\ \tau_n)}{d\tau} - \sum_{i=1}^{n-1}\bar{g}_i f_i^n\right]. \tag{26}$$

Relations (9)-(11), (16)-(18), and (26) are approximation solutions to the Volterra integral equations of the first kind. This problem belongs to the category of improperly formulated ones. One may expect, therefore, that the resulting systems of linear algebraic equations (the recurrence formulas represent systems with lower-order triangular matrices) will not be rational and their solutions will be unstable. Let us discuss the applicability of our results from the heuristic standpoint. At sufficiently small values of x_1, the expression for the transient thermal flux is

$$q(\tau) = -q_1 + \frac{\lambda}{a}\,x_1\frac{dT_1}{d\tau}.$$

Since this expression does not contain derivatives $d^nT/d\tau^n$ and $d^nq/d\tau^n$ of higher than the first order, hence calculations based on it will be relatively stable. An analogous result has been obtained in [7] for the case of a plate heated with a constant thermal flux $q = \text{const}$ at $\Delta \text{Fo} \geq 0.35$–$0.50$ (depending on x_1). If now the entire heating period is divided into i intervals and q_i = const is assumed on each, then we may postulate that for $\Delta\text{Fo} \geq 0.35$–$0.50$ the problem of determining q_i has been formulated rather rationally. Indeed, in these cases the singularities of function $q(\tau)$ are much less concealed and the solution to the forward problem becomes regularized:

$$\vartheta_i = \frac{b}{\lambda_0}\left(\text{Fo}_i + \frac{3(b-x_1)^2 - b^2}{6b^2}\right).$$

From here we have

$$\theta(x, \tau) = \int_0^\tau q(\xi)\frac{\partial\vartheta(x, \tau-\xi)}{\partial\tau}\, d\xi \simeq \frac{a}{\lambda_0 b}\sum_{i=1}^{n} \bar{q}_i(\tau_i - \tau_{i-1}).$$

A similar relation applies also to a semiinfinitely large body [3].

For specific problems, stable approximations can be realized at much lower values of ΔFo. The decisive factor here is the consistency of the input data and the distance of the point with a known temperature from the body boundary. An experiment must be simulated on the computer for every case.

It is to be noted that, with our approach to solving reverse heat-conduction problems, a formal improvement of the accuracy of approximating the unknown functions $q(\tau)$ and $\nu(\tau)$ will make it feasible to increase the time steps as the approximate limit of a stable solution is approached. For instance, the critical time interval $\Delta\tau_{cr}$ will be larger with formula (11) than with formula (10).

On the whole, the described schemes for determining the boundary conditions are useful in cases where the temperature input data are rather accurate, the temperature probes are located close to the body boundary, and the boundary conditions need not be defined very precisely. For other cases the solutions must be regularized [8, 9].

In the next article we will discuss direct numerical methods of solving nonlinear reverse heat-conduction problems and the effect which the quality of input data has on the accuracy and the stability of resulting approximations.

NOTATION

a is the thermal diffusivity;
b is the plate thickness;
C is the specific heat referred to volume;
g is an auxiliary function;
q is the thermal flux;
q_{in} is the thermal flux at the inner wall;
T is the temperature;
X is the space coordinate of the movable body boundary;
x is the local coordinate;
Fo is the Fourier number;
θ is the model temperature;
λ is the thermal conductivity;
ν is the thermal potential density;
τ is the time.

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